Formal Theory Exam

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You have to answer Questions I and II. Then you can choose one of Questions III and IV. That is, you have to answer three questions in total.

Question I. Fundamentals of Political Competition

1. State carefully the Hotelling-Downs median voter theorem, and sketch a proof of the theorem.

2. Consider a society where income y is distributed according to a CDF F, whose mean is μ and whose median is m. For each individual, income is fixed. Citizens care about a private good x, whose price is one, and a public good G. G is measured in units of per capita cost. (Thus, if $G = 100$, this means that the level of the public good is that which can be provided if society contributes \$100 per capita towards funding it.) The utility function of every citizen is

$$u(x, G) = x - \alpha(G - \mu)^2.$$

The *policy* is a constant tax rate t on income, where $t \in [0, 1]$.

Write down the indirect utility function $v(t; y)$ of a citizen of income y.

3. Compute the ideal tax policy for a citizen whose income is y.

4. Does the median voter theorem apply in this example?

5. Now suppose there are two political parties, and they have preferences of two citizens of incomes y_1 and y_2, where $y_1 < m < y_2$, (i.e., Party 1 represents a relatively poor voter and Party 2 represents a relatively rich voter). Suppose now that there is uncertainty as to which party will win the election if the two parties propose tax rates t_1 and t_2. Suppose this uncertainty is described by the *error distribution model with parameter* β. Let $t_1 > t_2$.

Write down the function $\pi(t_1, t_2)$ which gives the probability that policy t_1 will defeat policy t_2.

6. For the political economy just described in part 5, we define a *political equilibrium* as a Nash equilibrium between Parties 1 and 2, where each partys payoff function is the expected utility of its representative constituent. Prove that, if the median ideal policy t^m lies strictly between 0 and 1, it is never a political equilibrium for both parties to propose t^m.

7. Consider the US presidential election which will occur in November. How might you go about estimating the parameter β?

Question II. Game Theory

A bridge fell during rush hour in a city. The cause of the disaster could be either mismanagement by the incumbent mayor of the city or simply random factors in nature. It is commonly known that each of the possibilities could come true with probability $\frac{1}{2}$. Only the mayor knows for sure whether it was her fault or not. The city legislature controlled by the mayor's rival party has asked for the mayor to implement a political measure on some other issue that is favorable to the opposition, threatening that if the mayor rejects the measure, the legislature will call for a formal investigation on the bridge disaster.

The mayor must first decide whether to approve or reject the opposition's proposal. Once she decides to approve the proposal, the bridge situation will be resolved without incurring any cost, but the measure will incur some electoral cost for the mayor: once the measure is implemented, the mayor's reelection probability will be 0.4, and the probability of the opposition's winning the mayoral election will be 0.6.

If the mayor rejects the measure, then the opposition must decide whether to call for an investigation or not. If the opposition does not call for it, the election probability for each party will be 0.5. On the other hand, once the investigation is made, the truth is revealed and everyone learns whether the disaster is the mayor's fault or not. If it turns out that the mayor is innocent, then the election probabilities for the mayor and for the opposition are 0.6 and 0.4, respectively. If it turns out that the mayor is responsible for the disaster, then the election probabilities for the mayor and for the opposition are 0.2 and 0.8, respectively.

The mayor and the opposition both care only about their own (expected) probability of winning in the election.

1. Write out an extensive form game (i.e., a game tree) that represents this strategic situation.

2. Define the set of strategies for the mayor and the opposition of the game.

3. Find all perfect Bayesian equilibria in both pure and mixed strategies.

Question III. Comparative Politics

(*Majoritarian vs Proportional Elections*) Consider a model in which there are three equal-sized regions, $J = 1, 2, 3$. Some voters are employed, while others are unemployed. The post-tax consumption of employed individuals is $c = (1-\tau)y$, where τ is a non-distorting tax and y is income. (Income and thus consumption are assumed to be the same for all employed individuals). Unemployed individuals, on the other hand, receive an unemployment subsidy, f.

Let η^k denote the probability that an individual of type k is employed. There are K different types: $k = 1, 2, \ldots, K$, and each type forms a continuum with equal mass; the total population has mass one. The average value of η^k in the population is η, which also denotes the fraction of employed individuals.

Individuals get utility from post-tax income or subsidies and from consumption of a local public good, the per capita value of which in region J is given by g^J. An individual of type k residing in region J therefore has the following indirect utility function:

$$V^{k,J}(\mathbf{q}) = \eta^k U(c) + (1-\eta^k)U(f) + H(g^J) \tag{1}$$

where $U(\cdot)$ and $H(\cdot)$ are both concave utility functions satisfying the usual (classical) conditions and $\mathbf{q} = [\tau, f, \{g^J\}]$ is the policy vector.

Summing over types k, the government budget constraint is therefore:

$$\eta y \tau = (1-\eta)f + \frac{1}{3}\sum_J g^J \tag{2}$$

Finally, there are two parties, $P = A, B$. We will consider a simple game in which parties simultaneously and non-cooperatively announce binding policy platforms, individuals vote, and the platforms are implemented. Both parties maximize their probability of winning office. An individual will vote for party A if and only if:

$$V^{k,J}(\mathbf{q}_A) > V^{k,J}(\mathbf{q}_B) + \sigma^{kJ} + \delta \tag{3}$$

Thus, σ^{kJ} measures the idiosyncratic "ideological" preference in favor of party B on the part of voters of type k in region J, while δ is an aggregate random shock in favor of party B (reflecting, perhaps, the vicissitudes of turnout due to election-day conditions, etc.). Within type k in region J, the parameter σ^{kJ} is distributed uniformly on $[\frac{-1}{2\phi^J} + \alpha^J, \frac{1}{2\phi^J} + \alpha^J]$. Therefore, the density ϕ^J varies across regions but is constant within types (i.e., we are focusing on heterogeneity of "ideological" preferences across regions). The parameter α^J gives the region-specific mean, with $\alpha^1 < \alpha^2 = 0 < \alpha^3$. We assume $\phi^2 > \phi^1 > \phi^3$, and $\alpha^1\phi^1 + \alpha^3\phi^3 = 0$.[1] Finally, the aggregate shock δ is distributed uniformly on $[\frac{-1}{2\psi}, \frac{1}{2\psi}]$.

In the questions that follow, we will compare policy under "majoritarian" and "proportional" electoral rules. Here, majoritarian elections imply that separate elections are held in the three regions, which coincide with voting districts; a party must win in two of the three regions to win office. Proportional elections, on the other hand, here imply that elections are held in a single nationwide district, and the party with the biggest vote share wins office (so "proportional" is somewhat of a misnomer; majoritarian here implies a smaller district magnitude than PR elections).

1. Formally define the "swing voter" of type k in region J. Use this expression to find the vote share of party A among voters of type k in region J.

2. Use your answer in part 1 to define the probability that party A wins office under proportional elections, where this probability is a function of the policy choices of party A (taking the choices of party B as given).

3. Solve for the equilibrium policies $\mathbf{q}_A^{PR} = \mathbf{q}_B^{PR} = [\tau_{PR}^*, f_{PR}^*, \{g_{PR}^{J*}\}]$, with "PR" indicating these are the policies proposed under proportional rules. (Remember to use the budget constraint to pin down the tradeoffs between the various components of the policy vector, from the point of view of the parties). Why is it the case that $\mathbf{q}_A^{PR} = \mathbf{q}_B^{PR}$?

4. Which district gets the most local public goods under PR? Why?

5. Now develop an expression for the objective function of party A under majoritarian elections. Again, solve for the equilibrium policies. Which district gets the most local public goods under majoritarian elections? Why?

6. Compare equilibrium public goods provision under majoritarian and proportional elections. Under which electoral system are more public goods provided?

7. What underlying mechanism in the model leads to this result? To what extent is this result consistent with empirical evidence (e.g. as provided in Persson and Tabellini 2000, 2003)?

[1] In order to ensure existence of equilibrium, we also have to assume that the region-specific means are "far enough" apart. Assume that this condition is satisfied.

Question IV. Distributive Justice

Suppose there are two kinds of family in a society, called A and D families. A families are advantaged, and D families are disadvantaged. The number of A families equals the number of D families. A child born into an A family will, as an adult, earn an income of y_A, and a child born into a D family will earn an income of y_D, where $y_A > y_D$. Each family will have one child.

We are interested in using a veil of ignorance thought experiment for attempting to rectify the injustice of the birth lottery – that some children will be born to disadvantaged families and some to advantaged families. Assume that the vNM utility function over income of all people in this society (children, adults) is

$$V(y) = \frac{y^\theta}{\theta}$$

where θ is a constant in the interval $(-\infty, 1)$. (Note: If $\theta = 0$, then we take $V(y) = \log y$.)

Behind the veil of ignorance, there is a single decision maker, called the Soul. The Soul must choose a policy (T^A, T^D): if a child is born into an A family, then as an adult it will receive a transfer of T^A, and if it is born into a D family, as an adult it will receive a transfer of T^D. Of course, one of these transfers must be positive and the other negative, unless they are both zero, because the budget must balance.

The Soul will choose the policy that maximizes the expected utility of the representative child behind the veil of ignorance – before the child knows which kind of family it will be born into.

1. Write the budget constraint that the policy chosen by the Soul must satisfy.

2. Write the optimization problem that the Soul solves.

3. Show that, at the optimal policy, children are completely insured against the risk of being born into a D family.

We now alter the problem. Suppose that being born into an A family confers *two* kinds of advantage: the income advantage, as described above, but also an advantage in transforming income into welfare. Thus the *welfare* of the adult an A child becomes will be equal to $W = ay$, if y is his income, and the welfare of an adult a D child becomes will be $W = dy$, if y is his income, where $a > d$. Welfare is interpersonally comparable. In other words, A

adults not only earn more, but they have better opportunities for converting their income into welfare (because of having a superior education, etc.). We now interpret the vNM utility function as applying to lotteries over *welfare*: that is,

$$V(W) = \frac{W^\theta}{\theta}.$$

The Soul again chooses an optimal transfer policy (T^A, T^D).

4. Compute the optimal policy. Note that in general, there is now incomplete insurance against the risk of the birth lottery.

5. For what values of θ is the optimal transfer policy perverse?

6. What happens as θ approaches $-\infty$? What is the interpretation?

7. What do you conclude from this exercise?