Formal Theory Exam August 2008

Instructions: Do both Problems I and II, and choose *two* of Problems III – V.

I. A problem on Wittman equilibrium

There is an economy with two types of worker: fraction f of workers earn wage w_1 and fraction $1-f$ earn w_2, where $w_2 > w_1 > 1$. There is a public good, whose level is measured in the per capita cost of its provision. The financing of the public good is provided by a linear income tax at some rate *t* on wages. After-tax income is spent by workers on a private good. The utility function of a worker of type *w* over the private good (*x*) and the public good (*G)* is

$$u(x,G) = x + h(G),$$

where *h* is a strictly concave increasing function such that $h(0) = 0$.

(a) Write down the indirect utility function for a worker of type *w* over tax rates *t*, where $t \in [0,1]$. Compute the condition for a pair of tax rates (t_1,t_2) such that worker type w_1 prefers tax rate t_1 and worker type w_2 prefers tax rate t_2. Call such a pair of tax rates an *admissible* pair of tax rates.

(b) Suppose that uncertainty in voting is governed by the error-distribution model with an error margin of $\beta > 0$. Compute the condition on *f* and β such that, at any admissible pair of tax rates, the probability that each tax rate wins the election is strictly between zero and one.

We now assume that this condition holds.

(c) Write down the payoff functions for two Wittman parties, each of which represents one of the two worker types – that is, each party desires to maximize the expected utility of its constituency.

(d) Compute the endogenous party Wittman equilibrium. This will be a pair of tax rates expressed in terms of the function *h*. Finally, suppose $h(G) = \frac{\mu^2}{2} - \frac{1}{2}(\mu - G)^2$, where μ is the average wage, and calculate the equilibrium. Check that both tax rates are strictly between zero and one.

II. An extensive form game

A bridge fell during rush hour. The cause of the disaster could be either mismanagement by the incumbent mayor of the city or simply random factors in nature. It is commonly known that each of the possibilities could come true with probability 0.5. Only the mayor knows for sure whether it was his fault or not.

The city legislature, controlled by the mayor's rival party, has asked for the mayor to implement a political measure on some other issue that is favorable to the opposition, threatening that if he rejects the measure the legislature will call for a formal investigation on the bridge disaster.

The mayor must first decide whether to accept or refuse the opposition's proposal. Once he decides to agree to the proposal, the bridge situation will be resolved without incurring any cost, but the measure will incur some electoral cost for the mayor: once the measure is implemented, the mayor's reelection probability will be 0.4 and the probability of the opposition's winning the mayoral election will be 0.6.

If he rejects implementing the measure, the opposition must decide whether to call for an investigation or not. If the opposition does not call for an investigation, the election probability for each party will be 0.5. On the other hand, once the investigation is made, the truth will be revealed and everyone is informed whether the disaster is the mayor's fault or not. If it turns out that the mayor is innocent, then the election probabilities for the mayor and for the opposition are 0.6 and 0.4, respectively. If it turns out that the mayor is responsible for the disaster, then the election probabilities for the mayor and for the opposition are 0.2 and 0.8, respectively.

The mayor and the opposition both care only about their own (expected) probability of winning in the election.

1. Write out an extensive form game representing this strategic situation.

2. Define the set of strategies for the mayor and the opposition of the game.

3. Find all weak sequential equilibria (perfect Baeysian equilibria) in both pure and mixed strategies

Choose *two* of the following three problems.

III. Distributive Justice

A. There are two types in a society, Advantaged (A) and Disadvantaged (D): fraction f of the population is Disadvantaged. The A's have wages which are uniformly distributed on the interval $[\frac{1}{2},\frac{3}{2}]$; the D's have wages which are uniformly distributed on the interval $[0,1]$. We attribute the variation in wages *within* each type as due to differential effort.

Suppose that all individuals have a utility function $u(x,G) = x + h(G)$ over a private good (x) and a public good (G). The government (which is a benevolent dictatorship) is contemplating what linear tax rate to assess on incomes to fund the public good.

Suppose that $h(G) = \frac{\mu^2}{2} - \frac{1}{2}(\mu - G)^2$, where μ is the mean income (wage).

(a) If the government is *utilitarian,* compute the tax rate it should set. Call this tax rate t^U.

(b) If the government is *Rawlsian,* compute the tax rate it should set. Call this tax rate t^R.

(c) If the government wishes to equalize opportunities for welfare, compute the tax rate it should set. (You should first propose, in this context, what an equal-opportunity tax rate is. There may be several correct views on this.) Call this tax rate t^{EO}.

(d) What is the relationship among these three tax rates? Does this make sense, and if so, why?

B. There is a society in which half the people are Tall and half are Short. Being Tall or Short is a circumstance beyond the control of the individual. Tall people have advantages in the job market over short people; the wage of tall people is uniformly distributed on [2,6] and the wage of short people is uniformly distributed on [1, 5] – in the absence of any taxation.

(a) Suppose society wishes to equalize *opportunities for income*, and to do so, it will assess a tax rate of t on the incomes of all tall people, and it will transfer the proceeds equally to all short people. However, taxing the tall people has deleterious incentive affects on the training they choose to acquire. If the tax rate on their income is t then the distribution of wages among them will be uniform on the interval $[2-t, 6-t]$. Transferring lumpsums to the short people, however, does not affect their incentives.

Compute the tax rate that equalizes opportunities for income. (You need not compute the square root that appears in the result.)

(b) To assess the tax rate in part (a), the state must know who is tall and who is short. Suppose that citizens think that it is unfair to be differentially taxed depending on one's height, and so the state can only levy a tax rate of t on *all* incomes, and then transfer the average tax revenue as a lumpsum to all individuals. Of course, by levying a tax rate on the Short, this affects their incentives for training, and a tax rate of t will cause the wage distribution of the Short to be uniform on the interval $[1-t, 5-t]$.

Compute the equal-opportunity tax rate in this situation.

(c) Note that this problem shows the conflict between equalizing opportunities and 'privacy.' How do you feel about this matter? Think of other attributes that you might substitute for height, and study your views on the matter.

IV. Political Economy

There are three agents N={1,2,3} who play an infinite-horizon bargaining game over a set X of alternatives defined by

$$X=\left\{x=(x_1,x_2,x_3)\in\mathbb{R}^3_+ \,\middle|\, \sum_{i\in N} x_i = 1\right\}.$$

The timing of the bargaining game is as follows. In period t=0, (1) each agent i is selected as a proposer with probability 1/3; (2) The selected proposer makes a proposal, $x \in X$; (3) all agents simultaneously vote to either accept or reject. If a majority of the agents accept, the proposal passes and the game ends; otherwise, the game moves to period t+1.
For t=1,2,…, if a proposal has not passed prior to t, then the bargaining proceeds the same as in the above paragraph with the proviso that (1) if t=3k+1 for some k=0,1,2,…, then each agent who was not the proposer in t-1 is selected as a proposer with probability 1/2; (2) if t=3k+2 for some k=0,1,2,…, then the agent who was the proposer neither in t-2 nor in t-1 becomes the proposer for sure; (3) otherwise, each agent is selected with probability 1/3.
If an alternative x passes in period t, each agent i receives payoff $\delta^t x_i$, where $\delta \in (0,1)$ is the common discount factor. If no alternative ever passes, each i gets zero payoff.
A stationary strategy for a player in this game is a strategy that specifies the same action for every structurally equivalent subgame. (Consider two subgames, one beginning at history h and another beginning at h', say G(h) and G(h'). We say the two subgames are structurally equivalent if (1) the sequences of actions from h and h' are the same and (2) for each player, the preference over the set of full plays in G(h) is the same as the preference over the set of full plays in G(h'). For example, the subgame that commences at the beginning of period 0 is structurally equivalent to the subgame that commences in the beginning of period 3.)

1. Define the set of stationary strategies for an arbitrary agent *i*.

2. Identify the conditions for a profile of strategies to be a stationary subgame perfect equilibrium with stage-undominated voting strategies.

3. Find a stationary subgame perfect equilibrium with stage-undominated voting strategies.

V. Comparative Politics

V -- Part I

Some scholars allege that corruption -- defined as extraction of public funds by politicians for personal pecuniary gain -- can destabilize democracy. This question asks you to investigate this idea formally, using a standard probabilistic voting model.

Imagine a country with two regions, North and South. Overall population in the country is normalized to mass one, with proportions q^N and $q^S = 1 - q^N$ living in the North and South, respectively.

There are two political parties in the country, A and B, that compete for office in elections by proposing public policies. Public policies consist of a proportional tax $\tau \in (0,1)$ on income, local public goods g_N and g_S, and "rents" r, which only parties care about. Here, g_N and g_S are local public goods provided in the North and South regions, respectively.

The income of all citizens is normalized to one, so the indirect utility of a citizen living in region $J \in \{N,S\}$ is

$$U_J(\tau, g_J) = 1 - \tau + H(g_J),$$

where $H(\cdot)$ is a twice-differentiable concave function that satisfies the classical (Inada) conditions. The government budget constraint is thus given by $\tau = g_N + g_S + r$.

Denote the policies offered by each party as τ^P, g_N^P, g_S^P, and r^P, with $P \in \{A,B\}$. In elections, voter *i* in region *J* votes for party A if and only if

$$U_J(\tau^A, g_J^A) \geq U_J(\tau^B, g_J^B) + \sigma^i + \delta.$$

Here, σ^i measures voter *i*'s ideological propensity to vote for party B; in each region, this ideology variable is distributed uniformly on the region-specific distribution $[-\frac{1}{2\phi^J}, \frac{1}{2\phi^J}]$, with ϕ^J being the density of the distribution in region *J*. The parties know the distribution of this variable within each region at the time they announce their platforms. In particular, they know that $\phi^N > \phi^S$.

On the other hand, the random variable δ is an aggregate "popularity shock" in favor of party B, which is common across regions and is distributed uniformly on $[-\frac{1}{2\rho}, \frac{1}{2\rho}]$. The realization of this random variable is unknown to the parties at the time they announce their platforms. One interpretation of δ is that it reflects uncertainty at the time policies are announced over whether conditions on election day will favor party A or party B.

Politicians care only about the rents they can extract from public funds; however, only the party that is in office can extract rents. The expected utility of party P at the time it announces policies is therefore $p^P \cdot r^P$, where p^P is the probability that party *P* wins the election.

Consider initially a single-period democratic game. (In Part II below, we will extend this set-up to an infinite-horizon model). The timing of the game is as follows:

(1) Parties A and B simultaneously and non-cooperatively announce policies g_N^P, g_S^P, and r^P, with $P \in \{A,B\}$ and τ^P being determined residually from the government budget constraint;
(2) The value of δ is realized and elections are held;
(3) The announced policies are implemented by the winning party, and utilities are realized. In particular, the party *P* that won office receives the rents r^P announced in (1).

The equilibrium solution concept in the game between parties is subgame perfect Nash equilibrium.

Questions – V: Part I

(1) First, define the "swing voter" in regions *N* and *S* who is just indifferent between the two parties, given the announced platforms. Use these expressions to find the vote share of party A in each region. Then use q^N and q^S to define the aggregate vote share of party A, in the country as a whole.

(2) Use your answer to question (1) and the distribution of δ to define the probability p^A that party A wins the election, where the probability is a function of announced policies. It may be helpful to use the notation $\phi \equiv q^N \cdot \phi^N + q^S \cdot \phi^S$, where ϕ is the average density of individual ideology (across regions). Note also that $p^B = 1 - p^A$, where p^B is the probability that party B wins the election.

(3) (a) Solve for party A's optimal choice of g_N, g_S, and r, given party B's announced policies. (Recall that τ is determined residually from the government budget constraint; you should assume an interior solution, so that the optimal choice is $\tau^{A^*} < 1$). Assume that $q^N \cdot \phi^N < q^S \cdot \phi^S$ and show that more local public goods are provided in the South than in the North.

(b) Argue in one or two sentences that in equilibrium, party A's optimal choice of policies coincides with party B's. Note the implications of this observation for the equilibrium probabilities of victory, and use this to simplify the expression for equilibrium rents you derived in (a).

(c) How does the equilibrium level of rents depend on ρ, the density of the aggregate popularity shock? Recall that ρ is inversely related to the variance of the aggregate shock. Can you give some intuition to explain the relationship between ρ and the equilibrium level of rents?

(4) Show that the optimal policies (optimal from the perspective of the politicians) that you found in part (3) differ from the ideal policies for voters in region N as well as for voters in region S.

V -- Part II

Now consider an infinite-horizon version of the single-period game.

Here, each democratic electoral period has the same timing as the single-period game discussed in Part I; the actors, utilities, available policies, and technologies are as described above. Now, however, each democratic electoral period is followed by an inter-electoral period. During this inter-electoral period, the Northern group may decide to mount a coup against democracy, at an exogenous one-time cost of θ. If the Northern group stages a coup, it can set policy in the inter-electoral period and in every subsequent period at its unconstrained ideal point. (A coup thus brings an "absorbing state" of the infinite-horizon game). On the other hand, if the Northern group does not stage a coup, then during the inter-electoral period policy remains set at the vector announced by the winning party in the previous electoral period.

All actors discount the future at the common discount rate $\beta \in (0,1)$. The solution concept is Markov-perfect equilibrium, a subset of subgame perfect Nash equilibria in which actions and strategies may be conditioned only on the payoff-relevant state variables.

Question – V: Part II

(1) Argue in one or two sentences that in any Markov-perfect equilibrium, the platforms announced by the parties during electoral periods will coincide with the equilibrium platforms of the single-period game in Part I.

(2) Write down an inequality that gives the incentive-compatibility condition for a coup for the Northern group. In particular, the left-hand side of the inequality should give the infinite-horizon payoff to the Northern group of a coup, net of the one-time cost of a coup; the right-hand side of the inequality should give the infinite-horizon payoff of the Northern group when it does not stage a coup and continues to live under democracy. (Assume that if just indifferent between a coup and no coup, the Northern group does not stage a coup).

(3) Does the incidence of coups increase or decrease in ρ, the density of the electoral shock? How is it related to the equilibrium level of rents (e.g., corruption)? Give an intuition for this result.