Formal Theory Exam. Yale University. Dept of Political Science
Summer 2019

Please complete all four problems. Show your work.

Problem 1

There is an economy of workers where the production of the private good produces a *public bad,* which can be thought of as emissions of CO_2 , which reduce welfare for all. The utility of a worker/consumer is given by:

$$u(x,L,Y) = x - \frac{L^2}{2} - \frac{\beta}{n}Y ,$$

where x is her consumption of the private good, L is her labor supply, Y is the total production of the private good, and n is the (large) number of workers. The last term in the utility function reflects the negative carbon externality associated with production. β is a positive constant.

Worker i has a skill level s^i. Thus, if he supplies L^i units of raw labor, the *efficiency units of labor* he supplies to the firm are $E^i = s^i L^i$.

The firm's production function is $Y = E$; total output of the good equals the total efficiency units of labor employed in production.

Workers treat the total product Y as something they cannot influence by their individual behavior, because n is large. They treat the public bad as a negative externality (because of the emissions associated with it) over which they have no individual control.

The state will assess a tax rate of t on the firm's total value of output: thus if the price of output is normalized to one, then the firm pays a tax of tY . The purpose of the tax is to reduce production, thus addressing the negative externality it causes. The tax revenue is distributed equally to all workers. Thus a worker's income consists of labor income plus his per capita share of the tax revenue.

Denote by w the wage rate for one *efficiency unit* of labor. Thus, a worker $i,$ who supplies L^i units of labor to the firm receives a wage of ws^iL^i . Since we have assumed the price of output is one, w can be interpreted as the real wage.

The firm is assumed to employ labor to maximize its after-tax profits.

A. For a wage $w,$ deduce the labor supplies L^i of optimizing workers.

B. Compute the profit-maximizing condition for the firm, which gives an expression for $w.$

C. Setting the total supply of labor in efficiency units equal to the demand for efficiency units by the firm, compute the equilibrium value of *w*.

D. Now, write down the indirect utility function of worker *i*, $v^i(t)$. This expresses the final utility of the worker when the tax rate is *t* and everybody optimizes and markets clear. In other words, each *t* will generate an equilibrium and associated utilities for all workers.

E. Show that the ideal tax rate for every worker is strictly between 0 and 1.

F. What is the equilibrium tax rate if there is Downsian political competition over the tax rate? (Make sure you check a necessary condition.)

G. Suppose that all workers have the same skill level, so there is one ideal tax rate. What number does this tax rate approach as the number of workers becomes large?

H. Suppose that (t^L, t^R) is the endogenous party Wittman equilibrium for this model with uncertainty, and suppose that $t^L \neq t^R$. Deduce that the average tax rate, $\frac{t^L + t^R}{2}$ must be less than one-half. [This question does not require much computation if you do it correctly.]

Problem 2

There is a population consisting of Advantaged people and Disadvantaged people, where the degree of advantage is beyond the control of the individual. One-third of the population is Disadvantaged, and two-thirds is Advantaged.

There is an educational budget of *M* per capita. The wage-earning capacity of an Advantaged individual will be

$$w^A = x^{\frac{1}{2}} E^{\frac{1}{2}},$$

where *x* is the amount invested in the individual's education, and *E* is the effort that she expends. The wage-earning capacity of a Disadvantaged individual will be

$$w^D = (bx)^{\frac{1}{2}} E^{\frac{1}{2}},$$

where *x* is invested in her education, and *E* is the effort she expends, where $b < 1$.

The effort distribution of Advantaged types is uniform on the interval $[0, \frac{3}{2}]$ and the effort distribution of the Disadvantaged type is uniform on the interval $[0,1]$.

The problem is to study the optimal allocation of the educational budget into amounts to be spent on the education of the two types (x^A, x^D) given the budget constraint and several concepts of distributive justice.

A. Propose how an opportunity egalitarian might solve this problem. (You have some latitude on how you wish to define the opportunity egalitarian's objective. Try to keep it simple.) Compute the ratio of expenditures per capita on members of the two types, $\left(\frac{x^A}{x^D}\right)^{EOp}$.

B. Propose how a utilitarian would solve this problem. Compute the ratio of expenditures at the optimum on the two types, $\left(\frac{x^A}{x^D}\right)^{Util}$.
C. The value of *b* is smaller, the more disadvantaged is the Disadvantaged type. Compute what happens to the above optimal ratios as *b* approaches zero.
D. What is meant by 'utility monster?' Give an example.

Problem 3

Consider a political system characterized by the following sets of actors: the general population, a selectorate of size S, a winning coalition of size W, and the leader. The selectorate is the subset of the population whose members may become part of the leader's support base, whereas the winning coalition is the subset of the selectorate whose support the current leader needs to stay in power.

The incumbent leader survives in power as long as he is supported by a winning coalition of size W. A challenger, who would like to replace the leader, must gain the support of an alternative winning coalition (also of size W), of which at least one member must be a defector from the leader's winning coalition.

In order to maintain his hold on power, the incumbent leader offers each member of his winning coalition a reward $w_I \geq 0$. Denote the reward that the challenger promises to the members of his alternative winning coalition by $w_C \geq 0$. Assume that both the incumbent and the challenger must offer the same reward to all members of his winning coalition. In order to be credible, both rewards must satisfy a budget constraint: For all $j \in \{I, C\}$, $w_j W \leq R$, where R denotes the government's revenue.

But the challenger faces an additional commitment problem: Upon seizing power from the incumbent, he may prefer to replace some members of the winning coalition that brought him to power with others from the selectorate. In particular, assume that if the challenger gains power, he will then form his final winning coalition from the W members of the selectorate for whom he has the highest "affinity." To keep the analysis simple, suppose that the challenger's affinity A_i^C for any member i of the selectorate is drawn from the standard uniform distribution, with affinities realized only after the challenger takes power. A member of the winning coalition who considers defecting to the challenger therefore expects to become a member of the challenger's ultimate winning coalition only if he is among the Wmembers of the selectorate with the highest realization of the affinity parameter A_i^C, which occurs with probability W/S. By contrast, members of the incumbent's winning coalition know that they are among the W members of the selectorate for whom the incumbent has the highest affinity, as they are already in the winning coalition.

A. Formulate this setting as an extensive form game with imperfect information and draw the game tree.
B. Explain the concept of a "commitment problem" and illustrate it formally using this setting.
C. Solve for the perfect Bayesian equilibrium of this game.
D. Conduct comparative statics with respect to W and S and outline their political implications.

Problem 4

Consider the following model of war and bargaining. There are two countries, A and B. They have a dispute over a prize of size one (think of a territory whose area or economic value we normalized to one). Denote country A's share of a negotiated settlement by x and country B's share by $1 - x$. Thus country A would like x to be as close to 1 as possible; country B would like x to be as small as possible.

An alternative way of settling their dispute is by fighting a war. In that case, countries A and B suffer the cost c_A and c_B, respectively, with $0 < c_A < 1$ and $0 < c_B < 1$. Country A wins the war with probability p; country B wins it with probability $1 - p$. The eventual winner gets the entire prize but both countries suffer their respective cost of fighting.

A. State country A's expected payoff from fighting a war, country B's expected payoff from fighting a war, the set of negotiated settlements preferred to war by country A, the set of negotiated settlements preferred to war by country B, and the set of negotiated settlements preferred to war by both countries. Call the last term the "bargaining range" and prove that it is positive.
B. On a line segment of length one, mark each of the above quantities for parameter values $p = 1/2$, $c_A = 3/10$, and $c_B = 4/10$.

Consider now the following modification to this setting. Denote by p_A the probability that country A wins a war when it attacks first and by p_B the probability that country A wins when B attacks first. Assume that there is a first-strike advantage, $p_A \geq p_B$. That is, each side expects to be more likely to win a war when it attacks first (rather than when it is defending against an attack initiated by its opponent.)

C. Suppose country A attacks first. On a line segment that includes the values 0 and 1, mark country A's expected payoff from fighting a war, country B's expected payoff from fighting a war, the set of negotiated settlements preferred to war by country A, the set of negotiated settlements preferred to war by country B, and the set of negotiated settlements preferred to war by both countries. Use parameter values $p_A = 9/10$ and $p_B = 1/10$.
D. Suppose country B attacks first. On a line segment that includes the values 0 and 1, mark country A's expected payoff from fighting a war, country B's expected payoff from

fighting a war, the set of negotiated settlements preferred to war by country A, the set of negotiated settlements preferred to war by country B, and the set of negotiated settlements preferred to war by both countries. Again, use parameter values $p_A = 9/10$ and $p_B = 1/10$.

E. Assuming $p_A \geq p_B$, outline the conditions under which the "bargaining range" is positive and offer a political intuition behind your result.